Long-range interactions generated by random Lévy flights: Spin-flip and spin-exchange kinetic Ising model in two dimensions

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A $d=2$ dimensional kinetic Ising model that evolves by a combination of spin flips and spin exchanges is investigated. The spin flips satisfy detailed balance for the equilibrium state of the Ising model at temperature T while the spin exchanges are random Lévy flights of dimension $\sigma=1.5$. Our Monte Carlo (MC) simulations show that the steady state of this system displays a second-order phase transition as T is lowered. Comparing the critical fluctuations of the magnetization to those of an Ising model in which the interaction decays with distance as $r^{-3.5}$, we find that, within the resolution of the MC data, the critical exponents and the scaling functions of the two systems coincide. We argue that this coincidence indicates that a recent conjecture about the random Lévy flights generating long-range interaction of the form $V_{\text{eff}}(r)\sim r^{-d-\sigma}$ is valid not only in the spherical limit and in $d=1$ but also in $d=2$.

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I. INTRODUCTION

It has been suggested recently [1] that the effective interactions generated by a dynamical process can be deduced by examining the phase transitions occurring in the steady state of the system. The method is based on an extension of the universality hypothesis to nonequilibrium states. For an equilibrium system, universality means that the scaling properties near a critical point are determined by (i) the dimensionality of the system, (ii) the symmetry of the order parameter, and (iii) the range of interactions. Thus knowing the dimensionality and the symmetry, a measurement of the critical exponents yields information about the dominant interactions. Assuming that universality applies to non-equilibrium critical points as well, the critical properties of a nonequilibrium system provide us with the effective interactions.

The above line of argument has been used [1–3] to study effective interactions in a kinetic Ising model [4] that evolves by a combination of spin flips (Glauber dynamics [5]) and spin exchanges (Kawasaki dynamics [6]). The spin flips are produced by a heat bath of temperature T, i.e., their rates satisfy detailed balance for the equilibrium state of the Ising model at temperature T. Simultaneously with spin flips, random ($T=\infty$) spin exchanges take place. If the spin exchanges are between nearest-neighbor sites and their rate is significantly smaller than the rate of spin flips, then there is no rigorous evidence of ordering transition in $d=1$ dimension [4,7], and the nonequilibrium ordering in $d=2$ is in general of Ising type [7]. The conclusion one may draw from the above observations is that although the infrequent nearest-neighbor exchanges may generate further-neighbor couplings, they do not change the short-range, ferromagnetic nature of the dominant interactions. This conclusion is in agreement with rather general renormalization-group arguments [8] and similar conclusions about the irrelevance of small nonequilibrium perturbation can be drawn for a number of kinetic Ising models with a variety of competing dynamics [9–11]. Note, however, that some exceptions to our statement above have been reported. It remains to understand how to match the observation of both non-Ising critical behavior, and apparent ordering for $d=1$ in some related systems [12,13] with the more general picture presented here.

A different situation occurs when the random exchanges of spins are of infinite range [2,3]. In that case, ferromagnetic ordering occurs even in $d=1$, and the critical magnetization fluctuations are indistinguishable from those of an equilibrium Ising model with infinite-range interactions. Thus one may conclude that the long-range random exchanges transform the nearest-neighbor ferromagnetic couplings into infinite-range interactions. This conclusion is supported by studies of the $d=2$ model where the equilibrium Ising transition, that is present without the spin exchanges, becomes a mean-field transition as soon as the rate of spin exchanges is different from zero. Further support for the above conclusion comes from studies of such details as the crossover between the equilibrium Ising and the nonequilibrium mean-field behavior. The crossover exponent found in the MC simulations [3] is equal to the exponent describing the crossover between short- and infinite-range Ising behavior.

In between the short- and infinite-range regime, there are interactions which decay with distance as a power law. The generation of those interactions can again be investigated since ferromagnetic interactions of the form $r^{-d-\sigma}$ produce a critical behavior that is distinct from both the mean-field and the short-range limit provided $d/2<\sigma<2-\eta$ where η is the correlation exponent in the short-range system [14]. A recent study indicates [1] that power-law interactions can be generated if local dynamics (such as spin flips) that satisfies detailed balance for a short-range Hamiltonian is coupled to $T=\infty$ anomalous diffusion (Lévy-flight [15] exchanges of spins). More precisely, if the dimension of the Lévy flights producing the exchanges of spins is $0<\sigma<2$ then the effective interaction is of the form

$$V_{\text{eff}}(r) \sim r^{-d-\sigma} . \tag{1}$$

This result has been shown to be valid both for a $d=1$ kinetic Ising model and for the spherical limit of a time-dependent, n component Landau-Ginzburg model that is constructed as a combination of models A and B of critical dynamics [16].

The result (1) may be rather general since it can be derived [1] by assuming that, in the long-wavelength limit, the noise associated with a conserved dynamics (Lévy-flight exchanges) is negligible compared to the noise that is associated with a nonconserved dynamics (spin flips). Although this assumption is valid for fluctuations around an equilibrium state, nonequilibrium steady states are known to display unexpected features. Thus the generality of (1) remains to be tested and, in the present paper, we test it for a one-component, $d=2$ system. More specifically, we use MC methods to study a $d=2$ spin-flip and spin-exchange kinetic Ising model in which the spin exchanges take place by Lévy flights of dimension $\sigma=1.5$. Our result is that the effective interactions generated in the system can again be described by the expression (1).

II. THE MODEL

Consider a kinetic Ising model in which stochastic Ising variables $s_{\mathbf{n}}=\pm 1$ occupy the sites of an $L\times L$ periodic square lattice. The dynamics consists of flips and exchanges of spins with the two processes occurring independently. The spin-flip rate at site $\mathbf{n}$ is given by the expression [3,5]

$$w_{\mathbf{n}}^{(1)}=\frac{1}{\tau_1}\left[1-s_{\mathbf{n}}\tanh\left[K\sum_{l} s_{\mathbf{n}+1}\right]\right] , \tag{2}$$

where τ_1 just sets the time scale, K controls the temperature T of the spin-flip heat bath, and the sum is over the nearest-neighbor sites of $\mathbf{n}$. The spin flips alone would drive the system to the equilibrium state of the $d=2$ Ising model at temperature T with the nearest-neighbor coupling J related to K through $K=J/kT$.

The exchanges of spins take place between sites $\mathbf{n}$ and $\mathbf{n}'$ that are either in the same row or in the same column. The rate of exchanges is given by

$$w_{\mathbf{n},\mathbf{n}'}^{(2)}=\frac{A}{\tau_2|\mathbf{n}-\mathbf{n}'|^{1+\sigma}} , \tag{3}$$

where $A=2\sum_{i=1}^{\infty} i^{-1-\sigma}$. In the exchanges described by $w_{\mathbf{n},\mathbf{n}'}^{(2)}$, the spins move randomly either in the vertical or horizontal direction, and the probability of moving a distance l is proportional to $l^{-1-\sigma}$. This variable-step random walk is called Lévy flight [15] of dimension σ. Note that the exchanges are independent of energy, thus they can be viewed as a process generated by a contact to a $T=\infty$ heat bath.

We choose to investigate the case of Lévy-flight exchanges of dimension $\sigma=1.5$. This choice is somewhat arbitrary since the validity of Eq. (1) can be tested anywhere in the $d/2=1<\sigma<2-\eta=1.75$ range where the universal characteristics of the phase transition depend on the exponent of the power-law interaction [14]. The motivation for choosing $\sigma=1.5$ is to avoid crossover effects that may hamper numerical analysis near $\sigma=1$ (crossover to mean-field behavior) and near $\sigma=1.75$ (crossover to short-range behavior).

The description of the flip-and-exchange model is completed by noting that the frequency of spin-flip attempts was equal to the frequency of spin exchanges ($\tau_1=\tau_2$) in the simulations. The ratio τ_1/τ_2 is expected to be an irrelevant variable unless $\tau_1/\tau_2\to 0$ or $\tau_1/\tau_2\to\infty$. In those limits one expects crossovers to short-range or mean-field behavior, respectively.

The test of the validity of (1) means comparing the critical properties of the flip-and-exchange model to those of the equilibrium Ising model that has a long-range Hamiltonian of the form

$$H=\sum_{\mathbf{n},\mathbf{n}'} J_{\mathbf{n},\mathbf{n}'} s_{\mathbf{n}} s_{\mathbf{n}'} , \tag{4}$$

where $\mathbf{n}$ and $\mathbf{n}'$ are on the sites of an $L\times L$ periodic square lattice and

$$J_{\mathbf{n},\mathbf{n}'}=\frac{J_0}{|\mathbf{n}-\mathbf{n}'|^{1+\sigma}} . \tag{5}$$

The critical exponents for this model have been calculated [14] by renormalization-group methods with the results given as power series in $\theta=2\sigma-d>0$ ($\sigma\neq 2$, fixed) or $\Delta\sigma=\sigma-d/2>0$ (d fixed). The exponents we shall need below for comparison with the flip-and-exchange model are the correlation length exponent ν and the susceptibility exponent γ. Actually, γ and ν are related by the scaling law $\gamma=\nu(2-\eta)$, and since $2-\eta=\sigma$ at least up to order [14] $O(\theta^3)$ and perhaps to all orders of θ, the combination γ/ν that is needed below in the finite-size scaling of the magnetization fluctuations, $\langle M^2\rangle$, is "exactly" known to be $\gamma/\nu=\sigma=1.5$. The other exponent that appears in $\langle M^2\rangle$ is $1/\nu$ which is calculated from $\nu=\gamma/\sigma$ using the series for γ given explicitly [14] to order θ^2. The value $1/\nu\approx 0.95$ is, of course, less accurate than the value of γ/ν. A more detailed comparison of critical properties involves not only the critical exponents but the scaling functions as well. Thus we also compare the finite-size scaling functions for $\langle M^2\rangle$ of the equilibrium and nonequilibrium models. Since no analytical results are available for $\langle M^2\rangle$, except its asymptotics characterized by the critical exponents, we calculated $\langle M^2\rangle$ by MC simulations.

III. MONTE CARLO SIMULATIONS

For both the flip-and-exchange and the long-range Ising model, we simulated systems with linear sizes in the range $L=11-41$. The limit on the maximum system size we can investigate comes from the presence of long-range interactions. A MC step/spin for a long-range Ising model involves $O(L^2)$ operations and, consequently, a

MC step takes $O(L^4)$ operations. An extra difficulty is the critical slowing down. Cluster algorithms [17] are not efficient for long-range systems and, near the critical point, the relaxation time in a Metropolis algorithm is expected to scale with system size at least as L^2. Thus the number of operations needed to gather reliable data on the critical properties scales with L as L^6. With our present facilities, this high power of L limits us to considering systems with $L \leq 45$.

The simulations of both the flip-and-exchange and the long-range model were carried out following the same steps. First, the time evolution of the magnetization and of the energy (nearest-neighbor correlations in case of flip-and-exchange model) were monitored and a rough estimate of the relaxation time was obtained. Then the steady-state value of the magnetization fluctuations $\langle M^2 \rangle$ and of $\langle M^4 \rangle$ was measured as a function of T (the control parameter for the flip-and-exchange model is the spin-flip temperature that is called from now on as the temperature even though the thermodynamic temperature has no well-defined meaning in this case). Next we determined the critical temperature T_c by plotting $U(T)=1-\langle M^4 \rangle/(3\langle M^2 \rangle^2)$ (Figs. 1 and 2). In the limit of $L\rightarrow\infty$, one has [18] $U(T<T_c)=\frac{2}{3}$, $U(T>T_c)=0$, and $U(T_c)=U^*$ with $0<U^*<\frac{2}{3}$. Thus curves of $U(T)$ for various finite L-s are expected to intersect at $T\approx T_c$ and $U\approx U^*$. This indeed happens as can be seen from Figs. 1 and 2, and we can get rather accurate estimates of T_c for both the flip-and-exchange model, $T_c=2.60\pm0.04$ (in units of J) and the long-range Ising model $T_c=5.58\pm0.04$ in units of J_0 (the length measured in units of the lattice constant).

Once we had an estimate of T_c, we analyzed the $\langle M^2 \rangle$ data by assuming that, near $T\approx T_c$, $\langle M^2 \rangle$ obeyed finite-size scaling [2,19],

$$\langle M^2 \rangle \approx L^{2+\gamma/\nu}\Phi_{\pm}(\epsilon L^{1/\nu}) , \tag{6}$$

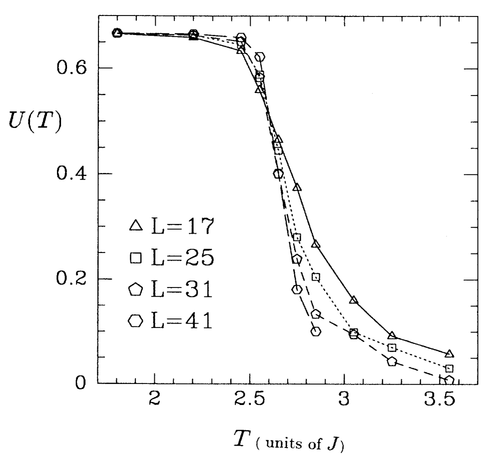

FIG. 1. Estimates of the critical temperature T_c from the intersections of $U(T)=1-\langle M^4 \rangle/(3\langle M^2 \rangle^2)$ curves for systems of varying linear sizes (L is measured in units of the lattice constant). The data are for the spin-and-exchange model with the temperature measured in units of the spin-flip coupling J. Curves are only meant as guides to the eyes.

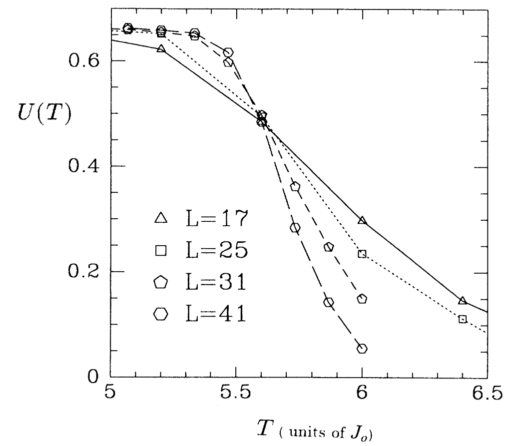

FIG. 2. The same as Fig. 1 but for the long-range Ising model. The temperature is measured in units of the coupling constant J_0.

where $\epsilon=|T-T_c|T_c$ and the two branches of the scaling function, Φ_+ and Φ_-, describe the ordered $(T<T_c)$ and disordered $(T>T_c)$ phases, respectively. We used γ and ν as fitting parameters and determined their values by best collapse of data when $\langle M^2 \rangle/L^{2+\gamma/\nu}$ was plotted against $\epsilon L^{1/\nu}$. Figure 3 shows the resulting scaling plot for the flip-and-exchange model when $\gamma/\nu=1.50$, and $1/\nu=1.12$ was chosen. One can see an excellent data collapse over two decades of the scaling variable $\epsilon L^{1/\nu}$. No perceptible decline in the quality of scaling can be observed if the exponents are varied within the intervals

$$\frac{\gamma}{\nu}=1.50\pm0.05 \tag{7}$$

and

$$\nu=1.12\pm0.10 . \tag{8}$$

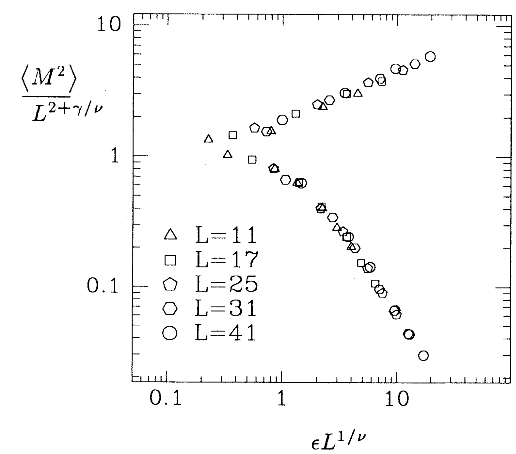

FIG. 3. Finite-size scaling of the magnetization fluctuations $\langle M^2 \rangle$ in the steady state of the flip-and-exchange model. The linear size of the square lattice is L and the deviations from the critical point are given by $\epsilon=|T_c/-T|/T_c$. The collapse of data was achieved by using $\gamma/\nu=1.5$, and $1/\nu=1.12$.

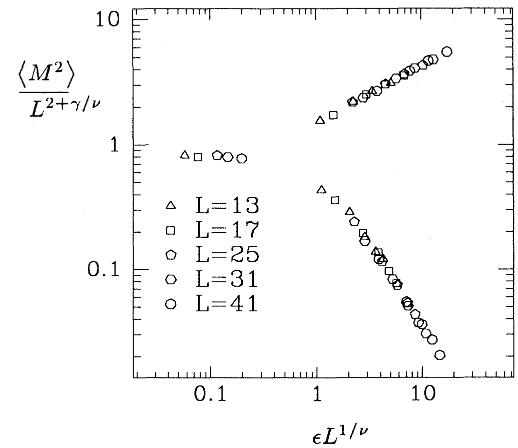

FIG. 4. The same as Fig. 3 but for the long-range Ising model. Scaling is observed for the same exponent values ($\gamma/\nu=1.5$, and $1/\nu=1.12$) as in Fig. 3.

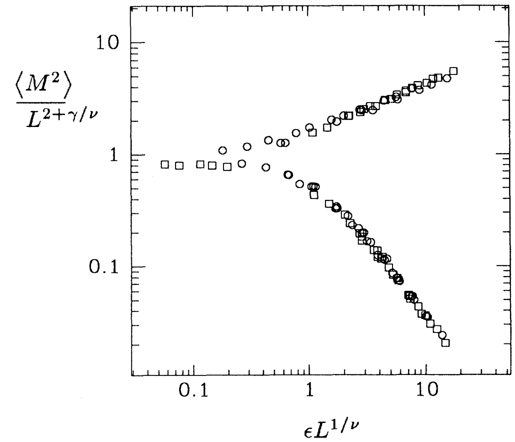

FIG. 5. Superposition of the scaling plots (Figs. 3 and 4) of $\langle M^2 \rangle$. The scales of ϵ and $\langle M^2 \rangle$ are unchanged for the flip-and-exchange model (○). However, all ϵ and $\langle M^2 \rangle$ values were multiplied by $\lambda_1=\lambda_2=0.905$ for the long-range Ising model (□). Note that these multiplications do not change the shape of the scaling function on the log-log plot. They just produce a uniform shift of the function.

The above value for γ/ν is in agreement with the exact renormalization-group results $\gamma/\nu=1.5$ for the long-range Ising model and the result for $1/\nu=1.12\pm0.10$ is also consistent with the second-order renormalization-group expansion ($1/\nu\approx0.96$). The difference between the MC and the renormalization-group values of $1/\nu$ may arise because an expansion to second order may well carry an error of 20%. Also, the MC estimates may be biased by systematic errors that are not included in the statistical error estimates.

In order to make the case of equal exponents more convincing, we made a finite-size scaling plot for $\langle M^2 \rangle$ of the long-range Ising model (Fig. 4). Using the same exponents as in the case of the flip-and-exchange model ($\gamma/\nu=1.5$ and $1/\nu=1.12$), we found again very good data collapse. Deterioration of the collapse was observed if the exponents values were taken out of the intervals given by Eqs. (7) and (8).

It is interesting that not only the critical exponents are equal but also the scaling functions seem to be the same in the two systems. Figure 5 shows the result of superposition of the $\langle M^2 \rangle$ data sets (Figs. 3 and 4). On this graph, we used two constants, λ_1 and λ_2, to relate the scales of ϵ and of $\langle M^2 \rangle$ in the two systems (i.e., $\lambda_1\epsilon$ and $\lambda_2\langle M^2 \rangle$ is plotted instead of ϵ and $\langle M^2 \rangle$ in case of the long-range Ising model). Note that scaling by λ_1 and λ_2 causes only a uniform shift of the curves but does not affect the shape of the scaling function. As can be seen from Fig. 5 where we used $\lambda_1=\lambda_2=0.905$, the overlap of the high-temperature (lower) branches of the scaling functions is excellent. The quality of overlap is worse for the low-temperature (upper) branches but we can still observe overlap within the simulation errors (the statistical error of the data points is of the order of the size of the symbols on the figure).

We believe that the results for the critical exponents and for the scaling functions, taken together, allow us the conclusion that the universal features of the order-parameter fluctuations in the two models are identical. This conclusion then leads to our final results: Lévy-flight exchanges of dimension $\sigma=1.5$ combined with spin flips satisfying detailed balance for the nearest-neighbor Ising model generate effective interactions of the form $V_{\text{eff}}(r)\sim r^{-3.5}$. This result is in accord with the suggestion that, in general, Lévy-flight exchanges of dimension σ generate an effective potential of the form $V(r)\sim r^{-d-\sigma}$.

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